

Integral closure of rings of integer-valued polynomials on algebras

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January 20, 2014

Abstract

Let D be an integrally closed domain with quotient field K . Let A be a torsion-free D -algebra that is finitely generated as a D -module. For every a in A we consider its minimal polynomial $\mu_a(X) \in D[X]$, i.e. the monic polynomial of least degree such that $\mu_a(a) = 0$. The ring $\text{Int}_K(A)$ consists of polynomials in $K[X]$ that send elements of A back to A under evaluation. If D has finite residue rings, we show that the integral closure of $\text{Int}_K(A)$ is the ring of polynomials in $K[X]$ which map the roots in an algebraic closure of K of all the $\mu_a(X)$, $a \in A$, into elements that are integral over D . The result is obtained by identifying A with a D -subalgebra of the matrix algebra $M_n(K)$ for some n and then considering polynomials which map a matrix to a matrix integral over D . We also obtain information about polynomially dense subsets of these rings of polynomials.

Keywords: Integer-valued polynomial, matrix, triangular matrix, integral closure, pullback, polynomially dense. MSC Primary: 13F20 Secondary: 13B25, 13B22 11C20.

1 Introduction

Let D be a (commutative) integral domain with quotient field K . The ring $\text{Int}(D)$ of integer-valued polynomials on D consists of polynomials in $K[X]$ that map elements of D back to D . More generally, if $E \subseteq K$, then one may define the ring $\text{Int}(E, D)$ of polynomials that map elements of E into D .

One focus of recent research ([4], [5], [9], [10], [11]) has been to generalize the notion of integer-valued polynomial to D -algebras. When $A \supseteq D$ is a torsion-free module finite D -algebra we define $\text{Int}_K(A) := \{f \in K[X] \mid f(A) \subseteq A\}$. The set $\text{Int}_K(A)$ forms a commutative ring. If we assume that $K \cap A = D$, then $\text{Int}_K(A)$ is contained in $\text{Int}(D)$ (these two facts are indeed equivalent), and often $\text{Int}_K(A)$ shares properties similar to those of $\text{Int}(D)$ (see the references above, especially [5]).

When $A = M_n(D)$, the ring of $n \times n$ matrices with entries in D , $\text{Int}_K(A)$ has proven to be particularly amenable to investigation. For instance, [9, Thm. 4.6] shows that the integral

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closure of $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ is $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$, where \mathcal{A}_n is the set of algebraic integers of degree at most n , and $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n) = \{f \in \mathbb{Q}[X] \mid f(\mathcal{A}_n) \subseteq \mathcal{A}_n\}$. In this paper, we will generalize this theorem and describe the integral closure of $\text{Int}_K(A)$ when D is an integrally closed domain with finite residue rings. Our description (Theorem 3.4) may be considered an extension of both [9, Thm. 4.6] and [2, Thm. IV.4.7] (the latter originally proved in [7, Prop. 2.2]), which states that if D is Noetherian and D' is its integral closure in K , then the integral closure of $\text{Int}(D)$ equals $\text{Int}(D, D') = \{f \in K[X] \mid f(D) \subset D'\}$.

Key to our work will be rings of polynomials that we dub *integral-valued polynomials*, and which act on certain subsets of $M_n(K)$. Let \overline{K} be an algebraic closure of K . We will establish a close connection between integral-valued polynomials and polynomials that act on elements of \overline{K} that are integral over D . We will also investigate polynomially dense subsets of rings of integral-valued polynomials.

In Section 2, we define what we mean by integral-valued polynomials, discuss when sets of such polynomials form a ring, and connect them to the integral elements of \overline{K} . In Section 3, we apply the results of Section 2 to $\text{Int}_K(A)$ and prove the aforementioned theorem about its integral closure. Section 4 covers polynomially dense subsets of rings of integral-valued polynomials. We close by posing several problems for further research.

2 Integral-valued polynomials

Throughout, we assume that D is an integrally closed domain with quotient field K . We denote by \overline{K} a fixed algebraic closure of K . When working in $M_n(K)$, we associate K with the scalar matrices, so that we may consider K (and D) to be subsets of $M_n(K)$.

For each matrix $M \in M_n(K)$, we let $\mu_M(X) \in K[X]$ denote the minimal polynomial of M , which is the monic generator of $N_{K[X]}(M) = \{f \in K[X] \mid f(M) = 0\}$, called the null ideal of M . We define Ω_M to be the set of eigenvalues of M considered as a matrix in $M_n(\overline{K})$, which are the roots of μ_M in \overline{K} . For a subset $S \subseteq M_n(K)$, we define $\Omega_S := \bigcup_{M \in S} \Omega_M$. Note that a matrix in $M_n(K)$ may have minimal polynomial in $D[X]$ even though the matrix itself is not in $M_n(D)$. A simple example is given by $\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \in M_2(K)$, where $q \in K \setminus D$.

Definition 2.1. We say that $M \in M_n(K)$ is *integral over D* (or just *integral*, or is an *integral matrix*) if M solves a monic polynomial in $D[X]$. A subset S of $M_n(K)$ is said to be *integral* if each $M \in S$ is integral over D .

Our first lemma gives equivalent definitions for a matrix to be integral.

Lemma 2.2. *Let $M \in M_n(K)$. The following are equivalent:*

- (i) M is integral over D
- (ii) $\mu_M \in D[X]$
- (iii) each $\alpha \in \Omega_M$ is integral over D

Proof. (i) \Rightarrow (iii) Suppose M solves a monic polynomial $f(X)$ with coefficients in D . As $\mu_M(X)$ divides $f(X)$, its roots are then also roots of $f(X)$. Hence, the elements of Ω_M are integral over D .

(iii) \Rightarrow (ii) The coefficients of $\mu_M \in K[X]$ are the elementary symmetric functions of its roots. Assuming (iii) holds, these roots are integral over D , hence the coefficients of μ_M are integral over D . Since D is integrally closed, we must have $\mu_M \in D[X]$.

(ii) \Rightarrow (i) Obvious. □

For the rest of this section, we will study polynomials in $K[X]$ that take values on sets of integral matrices. These are the integral-valued polynomials mentioned in the introduction.

Definition 2.3. Let $S \subseteq M_n(K)$. Let $K[S]$ denote the K -subalgebra of $M_n(K)$ generated by K and the elements of S . Define $S' := \{M \in K[S] \mid M \text{ is integral}\}$ and $\text{Int}_K(S, S') := \{f \in K[X] \mid f(S) \subseteq S'\}$. We call $\text{Int}_K(S, S')$ a set of *integral-valued polynomials*.

Remark 2.4. In the next lemma, we will prove that forming the set S' is a closure operation in the sense that $(S')' = S'$. We point out that this construction differs from the usual notion of integral closure in several ways. First, if S itself is not integral, then $S \not\subseteq S'$. Second, S' need not have a ring structure. Indeed, if $D = \mathbb{Z}$ and $S = M_2(\mathbb{Z})$, then both $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ are in S' , but neither their sum nor their product is integral. Lastly, even if S is a commutative ring then S' need not be the same as the integral closure of S in $K[S]$, because we insist that the elements of S satisfy a monic polynomial in $D[X]$ rather than $S[X]$.

However, if S is a commutative D -algebra and it is an integral subset of $M_n(K)$ then S' is equal to the integral closure of S in $K[S]$ (see Corollary 1 to Proposition 2 and Proposition 6 of [1, Chapt. V]).

Lemma 2.5. Let $S \subseteq M_n(K)$. Then, $(S')' = S'$.

Proof. We just need to show that $K[S'] = K[S]$. By definition, $S' \subseteq K[S]$, so $K[S'] \subseteq K[S]$. For the other containment, let $M \in K[S]$. Let $d \in D$ be a common multiple for all the denominators of the entries in M . Then, $dM \in S' \subseteq K[S']$. Since $1/d \in K$, we get $M \in K[S']$. □

An integral subset of $M_n(K)$ need not be closed under addition or multiplication, so at first glance it may not be clear that $\text{Int}_K(S, S')$ is closed under these operations. As we now show, $\text{Int}_K(S, S')$ is in fact a ring.

Proposition 2.6. Let $S \subseteq M_n(K)$. Then, $\text{Int}_K(S, S')$ is a ring, and if $D \subseteq S$, then $\text{Int}_K(S, S') \subseteq \text{Int}(D)$.

Proof. Let $M \in S$ and $f, g \in \text{Int}_K(S, S')$. Then, $f(M), g(M)$ are integral over D . By Corollary 2 after Proposition 4 of [1, Chap. V], the D -algebra generated by $f(M)$ and $g(M)$ is integral over D . So, $f(M) + g(M)$ and $f(M)g(M)$ are both integral over D and are both in $K[S]$. Thus, $f(M) + g(M), f(M)g(M) \in S'$ and $f + g, fg \in \text{Int}_K(S, S')$. Assuming $D \subseteq S$, let $f \in \text{Int}_K(S, S')$ and $d \in D$. Then, $f(d)$ is an integral element of K . Since D is integrally closed, $f(d) \in D$. Thus, $\text{Int}_K(S, S') \subseteq \text{Int}(D)$. □

We now begin to connect our rings of integral-valued polynomials to rings of polynomials that act on elements of \overline{K} that are integral over D . For each $n > 0$, let

$$\Lambda_n := \{\alpha \in \overline{K} \mid \alpha \text{ solves a monic polynomial in } D[X] \text{ of degree } n\}$$

In the special case $D = \mathbb{Z}$, we let $\mathcal{A}_n := \Lambda_n = \{\text{algebraic integers of degree at most } n\} \subset \overline{\mathbb{Q}}$.

For any subset \mathcal{E} of Λ_n , define

$$\text{Int}_K(\mathcal{E}, \Lambda_n) := \{f \in K[X] \mid f(\mathcal{E}) \subseteq \Lambda_n\}$$

to be the set of polynomials in $K[X]$ mapping elements of \mathcal{E} into Λ_n . If $\mathcal{E} = \Lambda_n$, then we write simply $\text{Int}_K(\Lambda_n)$. As with $\text{Int}_K(S, S')$, $\text{Int}_K(\mathcal{E}, \Lambda_n)$ is a ring despite the fact that Λ_n is not.

Proposition 2.7. *For any $\mathcal{E} \subseteq \Lambda_n$, $\text{Int}_K(\mathcal{E}, \Lambda_n)$ is a ring, and is integrally closed.*

Proof. Let Λ_∞ be the integral closure of D in \overline{K} . We set $\text{Int}_{\overline{K}}(\mathcal{E}, \Lambda_\infty) = \{f \in \overline{K}[X] \mid f(\mathcal{E}) \subseteq \Lambda_\infty\}$. Then, $\text{Int}_{\overline{K}}(\mathcal{E}, \Lambda_\infty)$ is a ring, and by [2, Prop. IV.4.1] it is integrally closed.

Let $\text{Int}_K(\mathcal{E}, \Lambda_\infty) = \{f \in K[X] \mid f(\mathcal{E}) \subseteq \Lambda_\infty\}$. Clearly, $\text{Int}_K(\mathcal{E}, \Lambda_n) \subseteq \text{Int}_K(\mathcal{E}, \Lambda_\infty)$. However, if $\alpha \in \mathcal{E}$ and $f \in K[X]$, then $[K(f(\alpha)) : K] \leq [K(\alpha) : K] \leq n$, so in fact $\text{Int}_K(\mathcal{E}, \Lambda_n) = \text{Int}_K(\mathcal{E}, \Lambda_\infty)$. Finally, since $\text{Int}_K(\mathcal{E}, \Lambda_\infty) = \text{Int}_{\overline{K}}(\mathcal{E}, \Lambda_\infty) \cap K[X]$ is the contraction of $\text{Int}_{\overline{K}}(\mathcal{E}, \Lambda_\infty)$ to $K[X]$, it is an integrally closed ring, proving the proposition. \square

Theorem 4.6 in [9] shows that the integral closure of $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ equals the ring $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$. As we shall see (Theorem 2.9), this is evidence of a broader connection between the rings of integral-valued polynomials $\text{Int}_K(S, S')$ and rings of polynomials that act on elements of Λ_n . The key to this connection is the observation contained in Lemma 2.2 that the eigenvalues of an integral matrix in $M_n(K)$ lie in Λ_n and also the well known fact that if $M \in M_n(K)$ and $f \in K[X]$, then the eigenvalues of $f(M)$ are exactly $f(\alpha)$, where α is an eigenvalue of M . More precisely, if $\chi_M(X) = \prod_{i=1, \dots, n} (X - \alpha_i)$ is the characteristic polynomial of M (the roots α_i are in \overline{K} and there may be repetitions), then the characteristic polynomial of $f(M)$ is $\chi_{f(M)}(X) = \prod_{i=1, \dots, n} (X - f(\alpha_i))$. Phrased in terms of our Ω -notation, we have:

$$\text{if } M \in M_n(K) \text{ and } f \in K[X], \text{ then } \Omega_{f(M)} = f(\Omega_M) = \{f(\alpha) \mid \alpha \in \Omega_M\}. \quad (2.8)$$

Using this fact and our previous work, we can equate $\text{Int}_K(S, S')$ with a ring of the form $\text{Int}_K(\mathcal{E}, \Lambda_n)$.

Theorem 2.9. *Let $S \subseteq M_n(K)$. Then, $\text{Int}_K(S, S') = \text{Int}_K(\Omega_S, \Lambda_n)$, and in particular $\text{Int}_K(S, S')$ is integrally closed.*

Proof. We first prove this for $S = \{M\}$. Using Lemma 2.2 and (2.8), for each $f \in K[X]$ we have:

$$f(M) \in S' \iff f(M) \text{ is integral} \iff \Omega_{f(M)} \subseteq \Lambda_n \iff f(\Omega_M) \subseteq \Lambda_n.$$

This proves that $\text{Int}_K(\{M\}, \{M\}') = \text{Int}_K(\Omega_M, \Lambda_n)$. For a general subset S of $M_n(K)$, we have

$$\text{Int}_K(S, S') = \bigcap_{M \in S} \text{Int}_K(\{M\}, S') = \bigcap_{M \in S} \text{Int}_K(\Omega_M, \Lambda_n) = \text{Int}_K(\Omega_S, \Lambda_n).$$

\square

The above proof shows that if a polynomial is integral-valued on a matrix, then it is also integral-valued on any other matrix with the same set of eigenvalues. Note that for a single integral matrix M we have these inclusions:

$$D \subset D[M] \subseteq \{M\}' \subset K[M].$$

Moreover, $\{M\}'$ is equal to the integral closure of $D[M]$ in $K[M]$ (because $D[M]$ is a commutative algebra).

3 The case of a D -algebra

We now use the results from Section 2 to gain information about $\text{Int}_K(A)$, where A is a D -algebra. In Theorem 3.4 below, we shall obtain a description of the integral closure of $\text{Int}_K(A)$.

As mentioned in the introduction, we assume that A is a torsion-free D -algebra that is finitely generated as a D -module. Let $B = A \otimes_D K$ be the extension of A to a K -algebra. Since A is a faithful D -module, B contains copies of D , A , and K . Furthermore, K is contained in the center of B , so we can evaluate polynomials in $K[X]$ at elements of B and define

$$\text{Int}_K(A) := \{f \in K[X] \mid f(A) \subseteq A\}.$$

Letting n be the vector space dimension of B over K , we also have an embedding $B \hookrightarrow M_n(K)$, $b \mapsto M_b$. More precisely, we may embed B into the ring of K -linear endomorphisms of B (which is isomorphic to $M_n(K)$) via the map $B \hookrightarrow \text{End}_K(B)$ sending $b \in B$ to the endomorphism $x \mapsto b \cdot x$. Consequently, starting with just D and A , we obtain a representation of A as a D -subalgebra of $M_n(K)$. Note that n may be less than the minimum number of generators of A as a D -module.

In light of the aforementioned matrix representation of B , several of the definitions and notations we defined in Section 2 will carry over to B . Since the concepts of minimal polynomial and eigenvalue are independent of the representation $B \hookrightarrow M_n(K)$, the following are well-defined:

- for all $b \in B$, $\mu_b(X) \in K[X]$ is the minimal polynomial of b . So, $\mu_b(X)$ is the monic polynomial of minimal degree in $K[X]$ that kills b . Equivalently, μ_b is the monic generator of the null ideal $N_{K[X]}(b)$ of b . This is the same as the minimal polynomial of $M_b \in M_n(K)$, since for all $f \in K[X]$ we have $f(M_b) = M_{f(b)}$. To ease the notation, from now on we will identify b with M_b .
- by the Cayley-Hamilton Theorem, $\deg(\mu_b) \leq n$, for all $b \in B$.
- for all $b \in B$, $\Omega_b = \{\text{roots of } \mu_b \text{ in } \overline{K}\}$. The elements of Ω_b are nothing else than the eigenvalues of b under any matrix representation $B \hookrightarrow M_n(K)$. If $S \subseteq B$, then $\Omega_S = \bigcup_{b \in S} \Omega_b$.
- $b \in B$ is *integral over D* (or just *integral*) if b solves a monic polynomial in $D[X]$.
- $B = K[A]$, since B is formed by extension of scalars from D to K .
- $A' = \{b \in B \mid b \text{ is integral}\}$. By [1, Theorem 1, Chapt. V] $A \subseteq A'$. In particular, this implies $A \cap K = D$ (because D is integrally closed), so that $\text{Int}_K(A) \subseteq \text{Int}(D)$ (see the remarks in the introduction).
- $\text{Int}_K(A, A') = \{f \in K[X] \mid f(A) \subseteq A'\}$.

Working exactly as in Proposition 2.6, we find that $\text{Int}_K(A, A')$ is another ring of integral-valued polynomials. Additionally, Lemma 2.2 and (2.8) hold for elements of B . Consequently, we have

Theorem 3.1. *$\text{Int}_K(A, A')$ is an integrally closed ring and is equal to $\text{Int}_K(\Omega_A, \Lambda_n)$.*

By generalizing results from [9], we will show that if D has finite residue rings, then $\text{Int}_K(A, A')$ is the integral closure of $\text{Int}_K(A)$. This establishes the analogue of [2, Thm. IV.4.7] (originally proved in [7, Prop. 2.2]) mentioned in the introduction.

We will actually prove a slightly stronger statement and give a description of $\text{Int}_K(A, A')$ as the integral closure of an intersection of pullbacks. Notice that

$$\bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \subseteq \text{Int}_K(A)$$

because if $f \in D[X] + \mu_a(X) \cdot K[X]$, then $f(a) \in D[a] \subseteq A$. We thus have a chain of inclusions

$$\bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \subseteq \text{Int}_K(A) \subseteq \text{Int}_K(A, A') \quad (3.2)$$

and our work below will show that this is actually a chain of integral ring extensions.

Lemma 3.3. *Let $f \in \text{Int}_K(A, A')$, and write $f(X) = g(X)/d$ for some $g \in D[X]$ and some nonzero $d \in D$. Then, for each $h \in D[X]$, $d^{n-1}h(f(X)) \in \bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X])$.*

Proof. Let $a \in A$. Since $f \in \text{Int}_K(A, A')$, $m := \mu_{f(a)} \in D[X]$, and $\deg(m) \leq n$.

Now, m is monic, so we can divide h by m to get $h(X) = q(X)m(X) + r(X)$, where $q, r \in D[X]$, and either $r = 0$ or $\deg(r) < n$. Then,

$$d^{n-1}h(f(X)) = d^{n-1}q(f(X))m(f(X)) + d^{n-1}r(f(X))$$

The polynomial $d^{n-1}q(f(X))m(f(X)) \in K[X]$ is divisible by $\mu_a(X)$ because $m(f(a)) = 0$. Since $\deg(r) < n$, $d^{n-1}r(f(X)) \in D[X]$. Thus, $d^{n-1}h(f(X)) \in D[X] + \mu_a(X) \cdot K[X]$, and since a was arbitrary the lemma is true. \square

For the next result, we need an additional assumption. Recall that a ring D has finite residue rings if for all proper nonzero ideal $I \subset D$, the residue ring D/I is finite. Clearly, this condition is equivalent to asking that for all nonzero $d \in D$, the residue ring D/dD is finite.

Theorem 3.4. *Assume that D has finite residue rings. Then, $\text{Int}_K(A, A') = \text{Int}_K(\Omega_A, \Lambda_n)$ is the integral closure of both $\bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X])$ and $\text{Int}_K(A)$.*

Proof. Let $R = \bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X])$. By (3.2), it suffices to prove that $\text{Int}_K(A, A')$ is the integral closure of R . Let $f(X) = g(X)/d \in \text{Int}_K(A, A')$. By Theorem 3.1, $\text{Int}_K(A, A')$ is integrally closed, so it is enough to find a monic polynomial $\phi \in D[X]$ such that $\phi(f(X)) \in R$.

Let $\mathcal{P} \subseteq D[X]$ be a set of monic residue representatives for $\{\mu_{f(a)}(X)\}_{a \in A}$ modulo $(d^{n-1})^2$.

Since D has finite residue rings, \mathcal{P} is finite. Let $\phi(X)$ be the product of all the polynomials in \mathcal{P} . Then, ϕ is monic and is in $D[X]$.

Fix $a \in A$ and let $m = \mu_{f(a)}$. There exists $p(X) \in \mathcal{P}$ such that $p(X)$ is equivalent to $m \bmod (d^{n-1})^2$, so $p(X) = m(X) + (d^{n-1})^2 r(X)$ for some $r \in D[X]$. Furthermore, $p(X)$ divides $\phi(X)$, so there exists $q(X) \in D[X]$ such that $\phi(X) = p(X)q(X)$. Thus,

$$\begin{aligned} \phi(f(X)) &= p(f(X))q(f(X)) \\ &= m(f(X))q(f(X)) + (d^{n-1})^2 r(f(X))q(f(X)) \\ &= m(f(X))q(f(X)) + d^{n-1}r(f(X)) \cdot d^{n-1}q(f(X)) \end{aligned}$$

As in Lemma 3.3, $m(f(X))q(f(X)) \in \mu_a(X) \cdot K[X]$ because $m(f(a)) = 0$. By Lemma 3.3, $d^{n-1}r(f(X))$ and $d^{n-1}q(f(X))$ are in $D[X] + \mu_a(X) \cdot K[X]$. Hence, $\phi(f(X)) \in D[X] + \mu_a(X) \cdot K[X]$, and since a was arbitrary, $\phi(f(X)) \in R$. \square

Theorem 3.4 says that the integral closure of $\text{Int}_K(A)$ is equal to the ring of polynomials in $K[X]$ which map the eigenvalues of all the elements $a \in A$ to integral elements over D .

Remark 3.5. By following essentially the same steps as in Lemma 3.3 and Theorem 3.4, one may prove that $\text{Int}_K(A, A')$ is the integral closure of $\text{Int}_K(A)$ without the use of the pullbacks $D[X] + \mu_a(X) \cdot K[X]$. However, employing the pullbacks gives a slightly stronger theorem without any additional difficulty.

In the case $A = M_n(D)$, $\text{Int}_K(M_n(D))$ is equal to the intersection of the pullbacks $D[X] + \mu_M(X) \cdot K[X]$, for $M \in M_n(D)$. Indeed, let $f \in \text{Int}_K(M_n(D))$ and $M \in M_n(D)$. By [11, Remark 2.1 & (3)], $\text{Int}_K(M_n(D))$ is equal to the intersection of the pullbacks $D[X] + \chi_M(X)K[X]$, for $M \in M_n(D)$, where $\chi_M(X)$ is the characteristic polynomial of M . By the Cayley-Hamilton Theorem, $\mu_M(X)$ divides $\chi_M(X)$ so that $f \in D[X] + \chi_M(X)K[X] \subseteq D[X] + \mu_M(X)K[X]$ and we are done.

Remark 3.6. The assumption that D has finite residue rings implies that D is Noetherian. Given that [2, Thm. IV.4.7] (or [7, Prop. 2.2]) requires only the assumption that D is Noetherian, it is fair to ask if Theorem 3.4 holds under the weaker condition that D is Noetherian.

Note that $\Omega_{M_n(D)} = \Lambda_n$ (and in particular, $\Omega_{M_n(\mathbb{Z})} = \mathcal{A}_n$). Hence, we obtain the following (which generalizes [9, Thm. 4.6]):

Corollary 3.7. *If D has finite residue rings, then the integral closure of $\text{Int}_K(M_n(D))$ is $\text{Int}_K(\Lambda_n)$.*

The algebra of upper triangular matrices yields another interesting example.

Corollary 3.8. *Assume that D has finite residue rings. For each $n > 0$, let $T_n(D)$ be the ring of $n \times n$ upper triangular matrices with entries in D . Then, the integral closure of $\text{Int}_K(T_n(D))$ equals $\text{Int}(D)$.*

Proof. For each $a \in T_n(D)$, μ_a splits completely and has roots in D , so $\Omega_{T_n(D)} = D$. Hence, the integral closure of $\text{Int}_K(T_n(D))$ is $\text{Int}_K(\Omega_{T_n(D)}, \Lambda_n) = \text{Int}_K(D, \Lambda_n)$. But, polynomials in $K[X]$ that move D into Λ_n actually move D into $\Lambda_n \cap K = D$. Thus, $\text{Int}_K(D, \Lambda_n) = \text{Int}(D)$. \square

Since $\text{Int}_K(T_n(D)) \subseteq \text{Int}_K(T_{n-1}(D))$ for all $n > 0$, the previous proposition proves that

$$\cdots \subseteq \text{Int}_K(T_n(D)) \subseteq \text{Int}_K(T_{n-1}(D)) \subseteq \cdots \subseteq \text{Int}(D)$$

is a chain of integral ring extensions.

4 Matrix rings and polynomially dense subsets

For any D -algebra A , we have $(A')' = A'$, so $\text{Int}_K(A')$ is integrally closed by Theorem 3.1. Furthermore, $\text{Int}_K(A')$ is always contained in $\text{Int}_K(A, A')$. One may then ask: when does $\text{Int}_K(A')$ equal $\text{Int}_K(A, A')$? In this section, we investigate this question and attempt to identify *polynomially dense* subsets of rings of integral-valued polynomials. The theory presented here is far from complete, so we raise several related questions worthy of future research.

Definition 4.1. Let $S \subseteq T \subseteq M_n(K)$. Define $\text{Int}_K(S, T) := \{f \in K[X] \mid f(S) \subseteq T\}$ and $\text{Int}_K(T) := \text{Int}_K(T, T)$. To say that S is *polynomially dense in T* means that $\text{Int}_K(S, T) = \text{Int}_K(T)$.

Thus, the question posed at the start of this section can be phrased as: is A polynomially dense in A' ?

In general, it is not clear how to produce polynomially dense subsets of A' , but we can describe some polynomially dense subsets of $M_n(D)'$.

Proposition 4.2. *For each $\Omega \subset \Lambda_n$ of cardinality at most n , choose $M \in M_n(D)'$ such that $\Omega_M = \Omega$. Let S be the set formed by such matrices. Then, S is polynomially dense in $M_n(D)'$. In particular, the set of companion matrices in $M_n(D)$ is polynomially dense in $M_n(D)'$.*

Proof. We know that $\text{Int}_K(M_n(D)') \subseteq \text{Int}_K(S, M_n(D)')$, so we must show that the other containment holds. Let $f \in \text{Int}_K(S, M_n(D)')$ and $N \in M_n(D)'$. Let $M \in S$ such that $\Omega_M = \Omega_N$. Then, $f(M)$ is integral, so by Lemma 2.2 and (2.8), $f(N)$ is also integral.

The proposition holds for the set of companion matrices because for any $\Omega \subset \Lambda_n$, we can find a companion matrix in $M_n(D)$ whose eigenvalues are the elements of Ω . \square

By the proposition, any subset of $M_n(D)$ containing the set of companion matrices is polynomially dense in $M_n(D)'$. In particular, $M_n(D)$ is polynomially dense in $M_n(D)'$.

When $D = \mathbb{Z}$, we can say more. In [10] it is shown that $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n) = \text{Int}_{\mathbb{Q}}(A_n, \mathcal{A}_n)$, where A_n is the set of algebraic integers of degree equal to n . Letting \mathcal{I} be the set of companion matrices in $M_n(\mathbb{Z})$ of irreducible polynomials, we have $\Omega_{\mathcal{I}} = A_n$. Hence, by Corollary 3.7 and Theorem 2.9, \mathcal{I} is polynomially dense in $M_n(\mathbb{Z})'$.

Returning to the case of a general D -algebra A , the following diagram summarizes the relationships among the various polynomial rings we have considered:

$$\begin{array}{ccccc} \text{Int}_K(A) \subseteq & \text{Int}_K(A, A') & = & \text{Int}_K(\Omega_A, \Lambda_n) & \\ & \cup & & \cup & \\ & \text{Int}_K(A') & = & \text{Int}_K(\Omega_{A'}, \Lambda_n) & (4.3) \\ & \cup & & \cup & \\ \text{Int}_K(M_n(D)') & = & \text{Int}_K(\Lambda_n) & & \end{array}$$

From this diagram, we deduce that A is polynomially dense in A' if and only if Ω_A is polynomially dense in $\Omega_{A'}$.

It is fair to ask what other relationships hold among these rings. We present several examples and a proposition concerning possible equalities in the diagram. Again, we point out that such equalities can be phrased in terms of polynomially dense subsets. First, we show that $\text{Int}_K(A)$ need not equal $\text{Int}_K(A, A')$ (that is, A need not be polynomially dense in A').

Example 4.4. Take $D = \mathbb{Z}$ and $A = \mathbb{Z}[\sqrt{-3}]$. Then, $A' = \mathbb{Z}[\theta]$, where $\theta = \frac{1+\sqrt{-3}}{2}$. The ring $\text{Int}_K(A, A')$ contains both $\text{Int}_K(A)$ and $\text{Int}_K(A')$.

If $\text{Int}_K(A, A')$ equalled $\text{Int}_K(A')$, then we would have $\text{Int}_K(A) \subseteq \text{Int}_K(A')$. However, this is not the case. Indeed, working mod 2, we see that for all $\alpha = a + b\sqrt{-3} \in A$, $\alpha^2 \equiv a^2 - 3b^2 \equiv a^2 + b^2$. So, $\alpha^2(\alpha^2 + 1)$ is always divisible by 2, and hence $\frac{\alpha^2(\alpha^2 + 1)}{2} \in \text{Int}_K(A)$. On the other hand, $\frac{\theta^2(\theta^2 + 1)}{2} = -\frac{1}{2}$, so $\frac{\alpha^2(\alpha^2 + 1)}{2} \notin \text{Int}_K(A')$. Thus, we conclude that $\text{Int}_K(A') \subsetneq \text{Int}_K(A, A')$.

The work in the previous example suggests the following proposition.

Proposition 4.5. *Assume that D has finite residue rings. Then, A is polynomially dense in A' if and only if $\text{Int}_K(A) \subseteq \text{Int}_K(A')$.*

Proof. This is similar to [2, Thm. IV.4.9]. If A is polynomially dense in A' , then $\text{Int}_K(A') = \text{Int}_K(A, A')$, and we are done because $\text{Int}_K(A, A')$ always contains $\text{Int}_K(A)$. Conversely, assume that $\text{Int}_K(A) \subseteq \text{Int}_K(A')$. Then, $\text{Int}_K(A) \subseteq \text{Int}_K(A') \subseteq \text{Int}_K(A, A')$. Since $\text{Int}_K(A')$ is integrally closed by Theorem 3.1 and $\text{Int}_K(A, A')$ is integral over $\text{Int}_K(A)$ by Theorem 3.4, we must have $\text{Int}_K(A') = \text{Int}_K(A, A')$. \square

By Proposition 4.2, $\text{Int}_K(M_n(D)') = \text{Int}_K(M_n(D), M_n(D)').$ There exist algebras other than matrix rings for which $\text{Int}_K(A') = \text{Int}_K(A, A')$. We now present two such examples.

Example 4.6. Let $A = T_n(D)$, the ring of $n \times n$ upper triangular matrices with entries in D . Define $T_n(K)$ similarly. Then, A' consists of the integral matrices in $T_n(K)$, and since D is integrally closed, such matrices must have diagonal entries in D . Thus, $\Omega_{A'} = D = \Omega_A$. It follows that $\text{Int}_K(T_n(D), T_n(D)') = \text{Int}_K(T_n(D)').$

Example 4.7. Let \mathbf{i}, \mathbf{j} , and \mathbf{k} be the standard quaternion units satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$ (see e.g. [8, Ex. 1.1, 1.13] or [3] for basic material on quaternions).

Let A be the \mathbb{Z} -algebra consisting of Hurwitz quaternions:

$$A = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid a_\ell \in \mathbb{Z} \text{ for all } \ell \text{ or } a_\ell \in \mathbb{Z} + \frac{1}{2} \text{ for all } \ell\}$$

Then, for B we have

$$B = \{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \mid q_\ell \in \mathbb{Q}\}$$

It is well known that the minimal polynomial of the element $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in B \setminus \mathbb{Q}$ is $\mu_q(X) = X^2 - 2q_0X + (q_0^2 + q_1^2 + q_2^2 + q_3^2)$, so A' is the set

$$A' = \{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in B \mid 2q_0, q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{Z}\}$$

As with the previous example, by (4.3), it is enough to prove that $\Omega_{A'} = \Omega_A$.

Let $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in A'$ and $N = q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{Z}$. Then, $2q_0 \in \mathbb{Z}$, so q_0 is either an integer or a half-integer. If $q_0 \in \mathbb{Z}$, then $q_1^2 + q_2^2 + q_3^2 = N - q_0^2 \in \mathbb{Z}$. It is known (see for instance [12, Lem. B p. 46]) that an integer which is the sum of three rational squares is a sum of three integer squares. Thus, there exist $a_1, a_2, a_3 \in \mathbb{Z}$ such that $a_1^2 + a_2^2 + a_3^2 = N - q_0^2$. Then, $q' = q_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is an element of A such that $\Omega_{q'} = \Omega_q$.

If q_0 is a half-integer, then $q_0 = \frac{t}{2}$ for some odd $t \in \mathbb{Z}$. In this case, $q_1^2 + q_2^2 + q_3^2 = \frac{4N - t^2}{4} = \frac{u}{4}$, where $u \equiv 3 \pmod{4}$. Clearing denominators, we get $(2q_1)^2 + (2q_2)^2 + (2q_3)^2 = u$. As before, there exist integers a_1, a_2 , and a_3 such that $a_1^2 + a_2^2 + a_3^2 = u$. But since $u \equiv 3 \pmod{4}$, each of the a_ℓ must be odd. So, $q' = (t + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})/2 \in A$ is such that $\Omega_{q'} = \Omega_q$.

It follows that $\Omega_{A'} = \Omega_A$ and thus that $\text{Int}_K(A, A') = \text{Int}_K(A')$.

Example 4.8. In contrast to the last example, the Lipschitz quaternions $A_1 = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$ (where we only allow $a_\ell \in \mathbb{Z}$) are not polynomially dense in A'_1 . With A as in Example 4.7, we have $A_1 \subset A$, and both rings have the same B , so $A'_1 = A'$. Our proof is identical to Example 4.4. Working mod 2, the only possible minimal polynomials for elements of $A_1 \setminus \mathbb{Z}$ are X^2 and $X^2 + 1$. It follows that $f(X) = \frac{x^2(x^2+1)}{2} \in \text{Int}_K(A_1)$. Let $\alpha = \frac{1+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2} \in A'$. Then, the minimal polynomial of α is $X^2 - X + 1$ (note that this minimal polynomial is shared by $\theta = \frac{1+\sqrt{-3}}{2}$ in Example 4.4). Just as in Example 4.4, $f(\alpha) = -\frac{1}{2}$, which is not in A' . Thus, $\text{Int}_K(A_1) \not\subseteq \text{Int}(A')$, so A_1 is not polynomially dense in $A'_1 = A'$ by Proposition 4.5.

5 Further questions

Here, we list more questions for further investigation.

Question 5.1. Under what conditions do we have equalities in (4.3)? In particular, what are necessary and sufficient conditions on A for A to be polynomially dense in A' ? In Examples 4.6 and 4.7, we exploited the fact that if $\Omega_A = \Omega_{A'}$, then $\text{Int}_K(A, A') = \text{Int}_K(A')$. It is natural to ask whether the converse holds. If $\text{Int}_K(A, A') = \text{Int}_K(A')$, does it follow that $\Omega_A = \Omega_{A'}$? In other words, if A is polynomially dense in A' , then is Ω_A equal to $\Omega_{A'}$?

Question 5.2. By [2, Proposition IV.4.1] it follows that $\text{Int}(D)$ is integrally closed if and only if D is integrally closed. By Theorem 3.1 we know that if $A = A'$, then $\text{Int}_K(A)$ is integrally closed. Do we have a converse? Namely, if $\text{Int}_K(A)$ is integrally closed, can we deduce that $A = A'$?

Question 5.3. In our proof (Theorem 3.4) that $\text{Int}_K(A, A')$ is the integral closure of $\text{Int}_K(A)$, we needed the assumption that D has finite residue rings. Is the theorem true without this assumption? In particular, is it true whenever D is Noetherian?

Question 5.4. When is $\text{Int}_K(A, A') = \text{Int}_K(\Omega_A, \Lambda_n)$ a Prüfer domain? When $D = \mathbb{Z}$, $\text{Int}_{\mathbb{Q}}(A, A')$ is always Prüfer by [9, Cor. 4.7]. On the other hand, even when $A = D$ is a Prüfer domain, $\text{Int}(D)$ need not be Prüfer (see [2, Sec. IV.4]).

Question 5.5. In Remark 3.5, we proved that $\text{Int}_K(M_n(D))$ equals an intersection of pullbacks:

$$\bigcap_{M \in M_n(D)} (D[X] + \mu_M(X) \cdot K[X]) = \text{Int}_K(M_n(D))$$

Does such an equality hold for other algebras?

Acknowledgement

The authors wish to thank the referee for his/her suggestions. The first author wishes to thank Daniel Smertnig for useful discussions during the preparation of this paper about integrality in non-commutative settings. The same author was supported by the Austrian Science Foundation (FWF), Project Number P23245-N18.

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